FREE ELECTRON LASER: OPERATING PRINCIPLES

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1. Introduction

Free Electron Laser (FEL) is one of the most recent among coherent radiation sources: in 1977 J.M.J Madey and coworkers at Stanford University obtained first lasing from an FEL operating at $\lambda = 3.417\ \mu m$, with $\Delta\lambda = 8$ nm bandwidth and an average power of 360 mW [1].
The FEL is really different from conventional lasers, because, instead of exploiting the stimulated emission from atomic or molecular systems, it makes use of the radiation emitted from a relativistic accelerated electron beam to obtain radiation amplification, through the interaction of the e-beam with a spatially periodic static magnetic field.
The first question that we must answer to is related to the possibility for a free electron to emit radiation without violating conservation principles for energy and momentum.
It is easy to notice that every time a "free" electron emits radiation, there must be the interaction with an external field, that allows the fulfillment of conservation rules. We can give some example of such fields:

- Synchrotron radiation emission: here it is the magnetic field of the bending magnets that allows conservation [2].
- Bremsstrahlung: the external field is the Coulomb field of the atomic nuclei [2]
- Compton scattering: in this case the E.M. field of the incident wave allows conservation [2]
- Smith-Purcell radiation: is the radiation generated by a charge passing close to a metal grid; in this case the field required for conservation is generated by the charges induced on the grid surface [3].
- Cherenkov radiation: is the radiation emitted by a charge moving in a medium with a velocity greater than the velocity of light in that medium; the field is generated by the asymmetric polarization induced in the dielectric [4]

1.1 FREE ELECTRON DEVICES

There are many devices used to produce radiation starting from Free Electron emission. Probably the most known Free Electron Device is the Klystron [5]. It is able to generate high power centimeter wavelength radiation. It was developed in the '30s by W.W. Hansen and coworkers and it was able to overcome the limitations of electronic tubes, whose ability to produce radiation was limited at high frequencies, when the triode dimensions were comparable to the wavelength and the cathode-anode flight time was no more negligible respect to the radiation oscillation period.

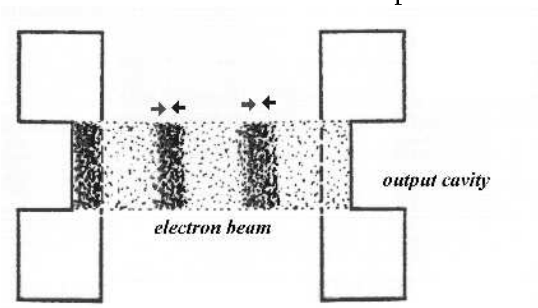

Fig. 1.1 : The klystron

In the Klystron (fig. 1.1) an accelerated electron beam is injected into a conductive cavity, whose dimensions are of the order of magnitude of the required emission wavelength. Electrons will find inside the cavity a Radio Frequency electric field, and its longitudinal component will produce a velocity modulation in the electron beam.
After a drift space this velocity modulation will result in a charge density modulation, thus producing the "bunching" of the electron beam. The bunched beam is then injected in another cavity, where it excites an E.M. wave. In this configuration the Klystron acts as an amplifier, but if we couple the second cavity to the first one, so that part of the generated radiation is given back to the first cavity to bunch the beam, we have realized an oscillator.
The operating principle of the Klystron, based upon the sequence velocity modulation → bunching → coherent emission, is very important, because it is common to many free electron devices.
There are many other Free Electron devices, like the Magnetron [5] and the Travelling Wave Tube (TWT) [5,6]. Other Free Electron Devices, realized in the '50s, are the Orotron [7], where something like a stimulated Smith-Purcell effect is used, the Ubitron [8], that can be considered a non-relativistic FEL, and the Gyrotron [9]. The limit of such devices is related to the shorter achievable wavelength, which is in the mm-wave region. Only with the FEL was possible to overcome this limitation, exploiting the relativistic effects of the high energy electron beam.

1.2 LORENTZ TRANSFORMATIONS

In order to better understand the FEL physics, it is necessary to consider the relativistic effects due to the high energy of the electron beam. We will then recall briefly the main equations that are used when considering a relativistic e-beam.
Let us consider two reference frames. We will call *x*, *y* and *z* the coordinates of the first system and with *x'*, *y'* and *z'* the coordinates of the second system. Time in the two systems will be denoted by *t* and *t'* respectively. Let us consider the second system moving respect to the first system at uniform velocity *v* along the *x* direction, as indicated in fig. 1.2

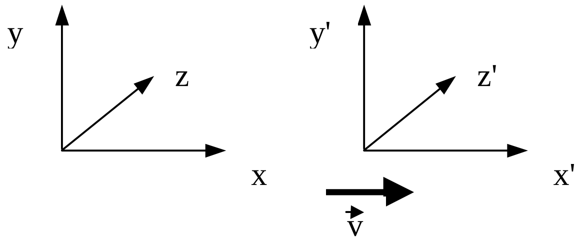

Fig. 1.2: Inertial reference frames

The coordinate transformation between the two systems are ruled by the Lorentz Transformations:

$$
\begin{aligned}
x' &= \gamma(x-vt) \\
y' &= y \\
z' &= z \\
t' &= \gamma(t-vx/c^2)
\end{aligned}
\qquad (1.1)
$$

where γ is the so-called “relativistic factor”:

$$\gamma = \frac{1}{\sqrt{1-v^2/c^2}} = \frac{1}{\sqrt{1-\beta^2}} \qquad (1.2)$$

Starting from these equations it is possible to describe the behavior of relativistic particles, when $\beta = v/c$ is close to 1.

1.3 DOPPLER EFFECT

If we want to derive the equations that describe the relation between the frequency ν in the first reference frame and the frequency ν' in the second reference frame, it is convenient to consider the light as a collection of photons. Energy and momentum of each photon can be expressed in terms of the frequency according to the following formulas:

$$\vec{p} = \frac{h\nu}{c}\vec{n}$$
$$E = h\nu \tag{1.3}$$

where $\vec{n}$ is the direction of propagation of the light wave and h=6.626•10^{-27} erg•s is the Plank constant.

It is now possible to introduce in our system the quadrivector formalism, i.e. an extension of the three-dimensional space coordinate vector, where a fourth coordinate is added in order to take into account the time transformations when changing reference frame in relativistic systems.

$$\boldsymbol{R} = (x, y, z, ct)$$

t was made dimensionally homogeneous with the spatial vector components by multiplying it by the constant c (velocity of the light).

We can then define the quadri-momentum as a quadrivector with the usual p_x, p_y e p_z space momentum component and the quantity E/c as time component.

$$\boldsymbol{P} = (p_x , p_y , p_z, E/c)$$

If we now apply the Lorentz transformations to the 4 components of the quadri-momentum:

$$p'_x = \gamma(p_x - \mathrm{v}t) = \gamma\left(p_x - \mathrm{v}\frac{E}{c^2}\right)$$
$$p'_y = p_y$$
$$p'_z = p_z \tag{1.4}$$
$$\frac{E'}{c^2} = \gamma\left(\frac{E}{c^2} - \mathrm{v}\frac{p_x}{c^2}\right) \Rightarrow E' = \gamma(E - \mathrm{v}\, p_x)$$

If we remind the expressions for E and p_x it is easy to obtain:

$$h\nu' = \gamma\left(h\nu - \mathrm{v}\frac{h\nu}{c}\cos\theta\right) \tag{1.5}$$

The angle θ is indicated in fig. 1.3

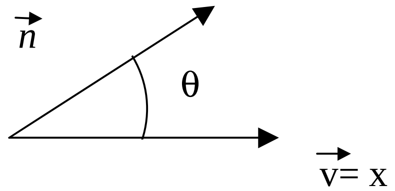

Fig. 1.3: coordinate system used for the frequency transformation

So there will be a relativistic Doppler shift, given by

$$\nu' = \gamma\nu(1-\beta\cos\theta) \tag{1.6}$$

if the propagation direction is along the x axis ($\theta = 0$) we have:

$$\nu' = \gamma\nu(1-\beta\cos\theta) = \nu(1-\beta)\frac{1}{\sqrt{1-\beta^2}} = \nu\sqrt{1-\beta}\,\frac{\sqrt{1-\beta}}{\sqrt{1-\beta^2}}\,\frac{\sqrt{1+\beta}}{\sqrt{1+\beta}} = \nu\,\frac{\sqrt{1-\beta}}{\sqrt{1+\beta}} \tag{1.7}$$

Using these formulas it is easy to understand why relativistic effects are important in a Free Electron Laser in order to decrease the emission wavelength.
In Free electron devices the emission wavelength is usually comparable to the physical dimensions of the structure producing the emission.
In the FEL emission is generated inside a magnetic undulator, i.e., in a spatially periodic magnetic field, that drives the electron beam into a spatial oscillation.
The system is sketched in fig. 1.4

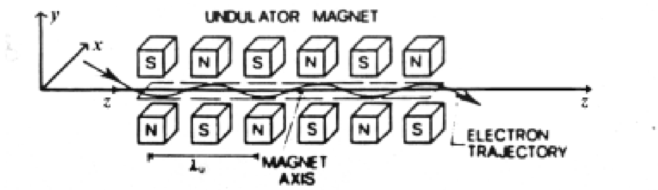

Fig. 1.4: coordinate system for electrons travelling inside a magnetic undulator [10]

If the electron beam is propagating along the *z* direction, due to the Lorentz Force electrons oscillate along the *x* direction, with a spatial period equal to the undulator period λ_u.
The associated frequency of oscillation is thus $\omega_u=2\pi v_z/\lambda_u \sim 2\pi c/\lambda_u$ for highly relativistic electron beams. If we consider a reference frame moving together with the electron at a longitudinal velocity v_z, in this system it will be possible to see the electron oscillate along the x direction, emitting at a frequency that will be upshifted due to Lorentz time transformation:

$$t'=\frac{t}{\gamma_z} \Rightarrow \omega'=\omega_u\gamma_z=\frac{\omega_u}{\sqrt{1-v_0^2/c^2}} \qquad \text{where } v_0=\langle v_z^2\rangle^{1/2} \tag{1.8}$$

In this reference frame the electron oscillate emitting like an antenna at frequency ω' all over the solid angle [2], and this frequency is upshifted by a factor γ. Moreover this is the frequency of emission in the reference frame moving with the electron, but we will see the radiation from the laboratory rest frame. So coming back to the rest frame, the emission will be compressed in a cone of aperture $\theta = 1/\gamma$ [2] and the frequency will experience a relativistic doppler shift given by:

$$\omega_0=\sqrt{\frac{1+\beta_0}{1-\beta_0}}\,\omega \quad \text{where} \quad \beta_0=\frac{v_0}{c}=\frac{\langle v_z^2\rangle^{1/2}}{c} \tag{1.9}$$

Substituting the expression for ω' and performing calculations we obtain:

$$\omega_0=\frac{2\gamma^2}{(1+K^2)}k_u c \qquad \text{where } k_u=2\pi/\lambda_u \tag{1.10}$$

that expressed in terms of wavelength gives:

$$\lambda_0=\frac{\lambda_u}{2\gamma^2}(1+K^2) \tag{1.11}$$

It is easy to notice that the dimension scale λ_u is reduced by a factor γ^2 exploiting the relativistic effects.

2. Synchrotron emission

It is well known that a charge emits radiation when accelerated. This behavior was already predicted in the XIX century: Larmor in 1897 derived, using classical electrodynamics, the formula expressing the power P irradiated by a non relativistic accelerated charge of mass m_0 and charge e :

$$P = \frac{2}{3}\frac{e^2}{m_0^2 c^3}\left|\frac{d\vec{p}}{dt}\right|^2 \qquad (2.1)$$

where $\vec{p}$ is the particle momentum and c the velocity of the light in vacuum.
This emission results to be important in circular particle accelerators: a relativistic electron of energy E = $m_0c^2\gamma$, moving at constant velocity $v=\beta c$ in a constant magnetic field B, due to the Lorentz Force will move along a circular trajectory of radius ρ :

$$\rho = \frac{E\beta}{Be} \approx \frac{E}{Be} \qquad \text{(for relativistic electrons } \beta \sim 1\text{)} \qquad (2.2)$$

The charge will then emit radiation and the irradiated power is:

$$P = \frac{2}{3}\frac{e^4 c^2}{\left(m_0 c^2\right)^4} E^2 B^2 \qquad (2.3)$$

This formula was derived in 1944 by Iwanenko e Pomeranchuk, and describes the power irradiated by a relativistic charge of energy E moving in a constant magnetic field B. It is easy to notice that this power is proportional to the square of the energy, but decreases as the 4^{th} power of the rest mass of the particle m_0. This means that we expect to notice a considerable amount of radiation emitted from low mass and high energy charges, i.e. from high energy electrons or positrons.
When building the first particle accelerators, this “synchrotron emission” was considered an annoying problem, because it was necessary to deliver continuously energy to the electron beam, in order to compensate the energy lost in irradiation.
Nevertheless the peculiar characteristics of such radiation appeared to be useful for many applications. The characteristics that make synchrotron radiation appealing are:

- Wide emission band, up to UV, x-ray e γ radiation
- tunability
- high intensity
- high polarization degree

The emission spectrum of synchrotron radiation is indicated in fig. 2.1

The so-called critical frequency ω_c is expressed by:

$$\omega_c = \frac{3}{2}\frac{c}{\rho}\left(\frac{E}{m_0c^2}\right)^3 = \frac{3}{2}\frac{c}{\rho}\gamma^3 \qquad (2.4)$$

where ρ is the bending radius expressed by (2.2) and γ is the relativistic factor :

$$\gamma = \frac{E}{m_0c^2} \qquad (2.5)$$

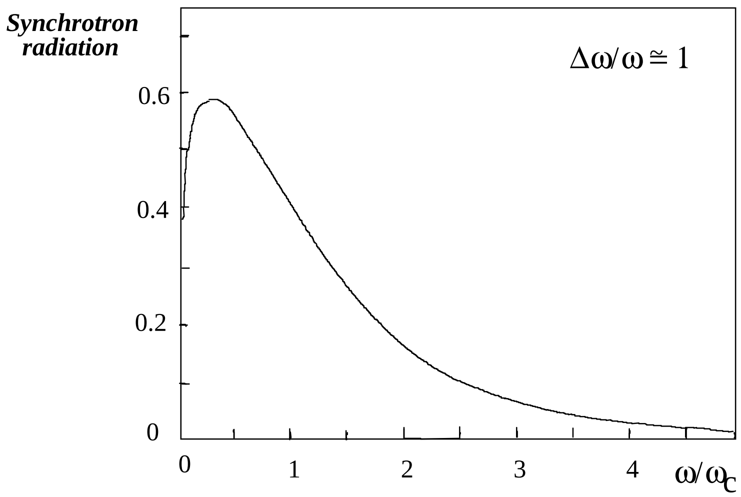

Fig. 2.1 – Synchrotron radiation spectrum

Looking at the figure it is possible to notice that a considerable amount of power is available up to frequency of about $2\omega_c$.
The value of ω_c depends on the 3rd power of the energy, so that increasing the electron energy, also the maximum frequency emitted with considerable power increases. As an example we can calculate the value of ω_c for 3 different kind of electron accelerators:

	ρ (m)	E (GeV)	ω_c (Hz)	λ_c
Small accelerator (Microtron)	1	0.02	$32 \bullet 10^{12}$	60 μm
Typical Synchrotron (Grenoble - Trieste)	40	2	$8 \bullet 10^{17}$	2.5 nm
Great accelerator (CERN)	1000	100	$4 \bullet 10^{21}$	$5 \bullet 10^{-4}$ nm

Table 2.1 : Critical frequencies calculated for different electron accelerators

In the first example the highest useful frequency is in the infrared region, while it is evident that the “typical” synchrotron is designed in order to obtain strong emission up to the x-ray region. The use of a much bigger machine allows strong emission at higher frequency, up to γ rays, but the use of such a machine is not suggested due to high costs and experimental difficulties.
Another interesting propriety of the synchrotron radiation is related to it directionality: due to the relativistic effects, radiation will not be emitted all around the solid angle, but it will be compressed in the forward direction, in a cone of aperture θ [2]. The light cone amplitude is expressed by:

$$\theta \cong \frac{m_0 c^2}{E} = \frac{1}{\gamma} \qquad (2.6)$$

For highly relativistic electrons the value of θ can be very small, giving rise to a highly directional emission. Let us calculate the value of θ for the previously examined machines:

	E (GeV)	θ
Small accelerator (Microtron)	0.02	25 mrad
Typical Synchrotron (Grenoble - Trieste)	2	250 μrad
Great accelerator (CERN)	100	5 μrad

Table 2.2 : Emission cone aperture calculated for different electron accelerators

Summarizing, increasing the electron energy, emitted power increases and concentrates along the motion direction.
The emission directionality can be used to explain the spectral characteristics of the synchrotron radiation: if we consider the light emitted by an electron moving along a circular trajectory of radius ρ, an hypothetical observer (fig. 2.2) will see the radiation emitted by the electron only when it’s position is inside the light cone, that is only along the arc $l_e = 2\theta\rho$.

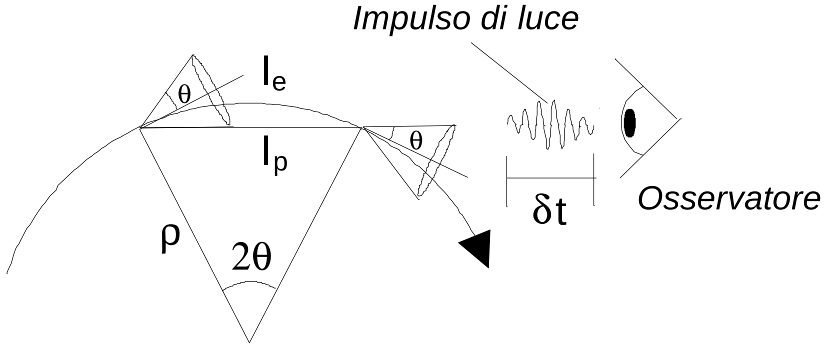

Fig. 2.2: Geometrical explanation of the spectral characteristics of synchrotron radiation

The pulse duration of the light pulse, as seen from the observer, will be equal to the different between the transit time of the electrons along the arc l_e and of the photon along the line $l_p = 2\rho \, sin\theta$. This difference can thus be written as:

$$\delta t = \frac{l_e}{v} - \frac{l_p}{c} \tag{2.7}$$

where v is the electron velocity, that can be expressed in terms of it's energy as:

$$v = c\sqrt{1-\left(\frac{m_0 c^2}{E}\right)^2} = c\sqrt{1-\frac{1}{\gamma^2}} \tag{2.8a}$$

Thus we obtain

$$\delta t = \frac{2\vartheta\rho}{c\sqrt{1-1/\gamma^2}} - \frac{2\rho \, sin\theta}{c} = \frac{2\rho}{c}\left\{\frac{\theta}{\sqrt{1-1/\gamma^2}} - sin\theta\right\} = \frac{2\rho}{c}\left\{\frac{\theta}{\sqrt{1-\theta^2}} - sin\theta\right\} \tag{2.8b}$$

if we remind that $\theta = 1/\gamma$. Expanding in power series up to first order in θ, i.e.

$$\frac{1}{\sqrt{1-\theta}} = 1 + \frac{1}{2}\theta + o(\theta^2) \qquad \text{and} \qquad sin\theta = \theta - \frac{1}{6}\theta^3 + o(\theta^5)$$, we obtain

$$\delta t = \frac{2\rho}{c}\left\{\theta\left[1+\frac{1}{2}\theta^2\right] - \left[\theta - \frac{1}{6}\theta^3\right]\right\} = \frac{4}{3}\frac{\rho}{c}\theta^3 = \frac{4}{3}\frac{\rho}{c}\frac{1}{\gamma^3} = \frac{4}{3}\frac{\rho}{c}\left(\frac{m_0 c^2}{E}\right)^3 \tag{2.8c}$$

According to Fourier analysis the bandwidth of a pulse of length δt is of the order of $\pi/\delta t$, so that we obtain:

$$\Delta\omega \approx \frac{\pi}{\delta t} = \frac{3}{4}\pi\frac{c}{\rho}\gamma^3 \tag{2.9}$$

that, except for a numerical factor 2π, is equal to the definition of the critical frequency, that is thus proportional to the bandwidth.
The small value of the θ angle for high energy electrons ($\theta \sim 1/\gamma$) will cause the emission of very short light pulses, that in turn will result in a wide emission band. Such a wide band is very useful, because it is possible to select the desired frequency in a wide range of possibilities, from the infrared region up to the x-ray region, giving rise to a wide tunability of the system. Nevertheless there is a drawback for such a situation: all the emitted power is distributed over a wide spectral range, so that power per unit

frequency results to be small. If our application requires high power per unit frequency, it is necessary to find a way to reduce the spectral bandwidth, i.e. to obtain longer light pulses. This can be done changing the interaction scheme for the electron, using the so-called "Motz Scheme” [11], using a magnetic undulator.

3. Undulator Emission

We have seen that synchrotron radiation, due to the wide spectral emission band, can provide a limited amount of power per unit frequency. In order to overcome this limit Motz proposed, in 1951, a new interaction scheme, designed to lengthen the duration of the light pulse emitted by an electron moving in a magnetic field.
In the “Motz Scheme” the electron moves inside a spatially periodic magnetic field, generated by a “magnetic undulator”. Scheme is sketched in fig. 3.1: under the effect of the almost sinusoidal magnetic field, the electron undergoes an oscillatory trajectory along the undulator axis.

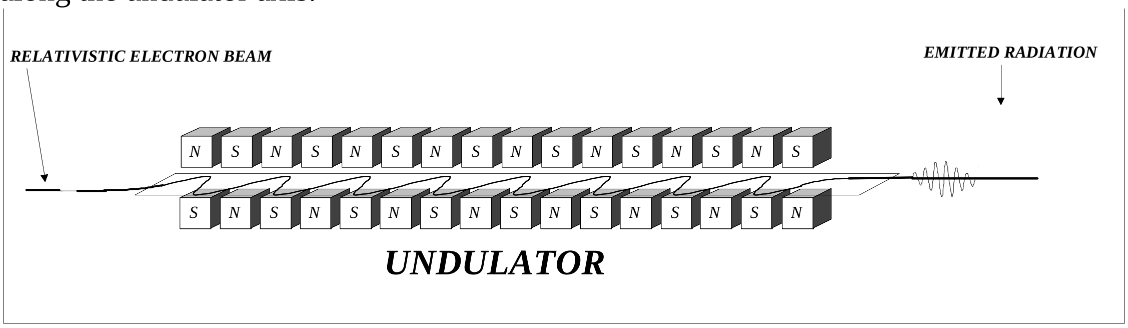

Fig. 3.1: Trajectory of electrons inside a magnetic undulator

If we choose the magnetic field intensity, the undulator period λ_u and the electron energy in such a way that the mean deviation angle from the undulator axis is smaller that the light emission cone θ, an hypothetical observer, placed along the undulator axis, would see the electron radiate all along the trajectory inside the undulator, thus generating a longer light pulse, that in turn means a narrower emission band.
When all these conditions are fulfilled, we can claim we are in the “undulator regime”
In order to derive the conditions for the “undulator regime” we can start from the motion equation for an electron inside a periodic static magnetic field. Using the reference frame in fig. 1.4 it is possible to write the expression for the Lorentz Force acting on an electron moving along the z axis:

$$\frac{d\vec{p}}{dt} = \frac{e}{c}\vec{v} \wedge \vec{B} \qquad (3.1)$$

where $\vec{B} = \left(0, B_0 \cos\left(\frac{2\pi}{\lambda_u} z\right), 0\right)$, λ_u is the undulator period and we have assumed a perfectly sinusoidal magnetic field (with only the *y* component different from zero).

In the vector product $\vec{v} \wedge \vec{B} = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ 0 & 0 & v_z \\ 0 & B_y & o \end{vmatrix}$ only the x component will be different from zero:

$$\vec{v} \wedge \vec{B} = -v_z B_0 \cos\left(\frac{2\pi}{\lambda_u} z\right)\hat{x} \tag{3.2}$$

This means that, due to the Lorentz Force, the electron will be accelerated along the transverse direction x, and that the acceleration will be described by:

$$\frac{dp_x}{dt} = \frac{e}{c}\frac{dz}{dt} B_0 \cos\left(\frac{2\pi}{\lambda_u} z\right) \tag{3.3}$$

performing a time integration:

$$p_x = \int \frac{dp_x}{dt} dt = \frac{eB_0}{c} \int \frac{dz}{dt} \cos\left(\frac{2\pi}{\lambda_u} z\right) dt = \frac{eB_0 \lambda_u}{2\pi\, c}\, sin\left(\frac{2\pi}{\lambda_u} z\right) \tag{3.4}$$

The mean deviation angle from the electron trajectory and the undulator axis is (with the reasonable approximation $p_z \sim p$):

$$\langle \theta \rangle = \left\langle \frac{(p_x)^2}{(p_z)^2} \right\rangle^{1/2} = \frac{eB_0 \lambda_u}{2\pi\, cp} \left\langle sin^2\left(\frac{2\pi}{\lambda_u} z\right)\right\rangle^{1/2} \tag{3.5a}$$

if we remind that $E \sim pc$, thus $p_z \sim E/c$ and that $<sin^2(x)> = ½$ we obtain:

$$\langle \theta \rangle = \frac{eB_0 \lambda_u}{2\sqrt{2}\ \pi E} \tag{3.5b}$$

If we want to obtain a "long" light pulse, this angle must be smaller than the emission cone aperture, that is expressed by $1/\gamma$; this can be written as follows:

$$\langle \theta \rangle = \frac{eB_0 \lambda_u}{2\sqrt{2}\ \pi E} \leq \frac{1}{\gamma} = \frac{m_0 c^2}{E} \tag{3.6}$$

writing the magnetic field as $B = B_0 \cos\left(\frac{2\pi}{\lambda_u} z\right)$ we obtain

$\langle B^2 \rangle = \frac{1}{2} B_0^2 \Rightarrow B_0 = \sqrt{2} \langle B^2 \rangle^{1/2}$, thus, to fulfil the "undulator regime" we must satisfy:

$$\frac{e\langle B^2\rangle^{1/2}\lambda_u}{2\pi m_0 c^2} \le 1 \tag{3.7a}$$

The quantity on the left is called "undulator parameter", and is denoted by K:

$$K = \frac{e\langle B^2\rangle^{1/2}\lambda_u}{2\pi m_0 c^2} \tag{3.8}$$

thus, we are in the "undulator regime" if:

$$K \le 1 \tag{3.7b}$$

When the undulator condition is fulfilled a long light pulse will be emitted, narrowing the spectral band, giving thus rise to emission characterized by higher power per unit frequency.
This increment can be easily calculated using a procedure similar to that used for synchrotron radiation emission. The time duration of the light pulse is given by the difference in transit time between the photons and the electrons in the undulator:

$$\delta t = \frac{L}{v_0} - \frac{L}{c} \tag{3.9}$$

where L is the undulator length and v_0 is the mean electron velocity along the undulator axis; in our reference frame this can be written as: $v_0 = \langle v_z^2\rangle^{1/2}$.
To obtain the value for $v_z^2 = v^2 - v_x^2$ we remind that:

$$\begin{cases} v^2 = c^2\left(1 - \dfrac{1}{\gamma^2}\right) \\ p_x = -\dfrac{eB_0\lambda_u}{2\pi c}\sin\left(\dfrac{2\pi}{\lambda_u}z\right) \text{ e } p = m_0 v\gamma \end{cases} \tag{3.10}$$

thus it is easy to obtain

$$v_x = -\frac{eB_0\lambda_u}{2\pi c m_0\gamma}\sin\left(\frac{2\pi}{\lambda_u}z\right) = -\frac{\sqrt{2}cK}{\gamma}\sin\left(\frac{2\pi}{\lambda_u}z\right) \tag{3.11}$$

then

$$v_z^2 = v^2 - v_x^2 = c^2\left[1 - \frac{1}{\gamma^2}\right] - c^2\left[\frac{2K^2}{\gamma^2}\sin^2\left(\frac{2\pi}{\lambda_u}z\right)\right] = c^2\left\{1 - \frac{1}{\gamma^2}\left[1 + 2K^2\sin^2\left(\frac{2\pi}{\lambda_u}z\right)\right]\right\} \tag{3.12}$$

averaging along an undulator period, and reminding that $\langle sin^2(x)\rangle = ½$ we obtain:

$$v_0^2 = \langle v_z^2 \rangle = c^2\left[1 - \frac{1}{\gamma^2}\left(1 + K^2\right)\right] \tag{3.13}$$

it is now possible to calculate δt:

$$\delta t = \frac{L}{v_0} - \frac{L}{c} = \frac{L}{c\sqrt{1-\left(1+K^2\right)/\gamma^2}} - \frac{L}{c}$$

(3.14)

It is known that $\frac{1}{\sqrt{1-x}} \xrightarrow[x\to 0]{} 1 + \frac{1}{2}x + o\left(x^2\right)$ and that for $\gamma >> 1$ $\frac{1}{\gamma} \to 0$, then

$$\delta t = \frac{L}{c}\left\{1 + \frac{1}{2\gamma^2}\left(1+K^2\right) - 1\right\} = \frac{L}{2c\gamma^2}\left(1+K^2\right) \tag{3.15}$$

The bandwidth expression is then:

$$\Delta\omega \approx \frac{\pi}{\delta t} = \frac{2\pi c}{\mathrm{L}}\frac{\gamma^2}{1+K^2} \tag{3.16}$$

It is possible to notice that for undulator emission the bandwidth $\Delta\omega$ is proportional to the square of the electron energy γ^2, while for circular synchrotrons it was proportional to the 3rd power of the energy γ^3; thus, for undulator emission, bandwidth is γ times narrower then for circular synchrotrons. When we deal with high energy electron, with $\gamma >> 1$, Motz Scheme allows a considerable narrowing of the emission bandwidth, and a corresponding increase of the power per unit frequency.

We already derived in chapter 1 the expression for the undulator emission; we repeat here, with some more details the derivation of the emission central frequency for a magnetic undulator.

If a relativistic electron beam is propagating along the z direction (undulator axis), due to the Lorentz Force electrons oscillate along the x direction, with a spatial period equal to the undulator period λ_u.

The associated frequency of oscillation is thus $\omega_u = 2\pi v_z/\lambda_u$; for highly relativistic electron beams $v_z \sim c$, thus:

$$\omega_u = \frac{2\pi}{\lambda_u} v_z \approx \frac{2\pi}{\lambda_u} c \tag{3.17}$$

If we consider a reference frame moving together with the electron at a longitudinal velocity v_z, in this system it will be possible to see the electron oscillate along the x direction, emitting at a frequency that will be upshifted due to Lorentz time transformation:

$$t' = \frac{t}{\gamma_z} \Rightarrow \omega' = \omega_u \gamma_z = \frac{\omega_u}{\sqrt{1 - v_0^2/c^2}} \qquad (3.18)$$

where $v_0 = \langle v_z^2 \rangle^{1/2}$

In this reference frame the electron oscillates emitting like an antenna at frequency ω' all over the solid angle [2], and this frequency is upshifted by a factor γ. Moreover this is the frequency of emission in the reference frame moving with the electron, but we will see the radiation from the laboratory rest frame. So coming back to the rest frame, the emission will be compressed in a cone of aperture $\theta = 1/\gamma$ and the frequency will experience a Relativistic Doppler Shift given by:

$$\omega_0 = \sqrt{\frac{1+\beta_0}{1-\beta_0}}\,\omega' \qquad (3.19a)$$

where $\beta_0 = \frac{v_0}{c} = \frac{\langle v_z^2 \rangle^{1/2}}{c}$; thus

$$\omega_0 = \sqrt{\frac{1+\beta_0}{1-\beta_0}}\,\omega' = \sqrt{\frac{1+\beta_0}{1-\beta_0}}\sqrt{\frac{1+\beta_0}{1+\beta_0}}\,\omega' = \frac{1+\beta_0}{\sqrt{1-\beta_0^2}}\,\omega' \qquad (3.19b)$$

reminding that $\omega' = \frac{\omega_u}{\sqrt{1-\beta_0^2}}$ we obtain:

$$\omega_0 = \frac{1+\beta_0}{\sqrt{1-\beta_0^2}}\frac{\omega_u}{\sqrt{1-\beta_0^2}} = \frac{1+\beta_0}{1-\beta_0^2}\,\omega_u \xrightarrow{\beta\to 1} \frac{2\omega_u}{1-\beta_0^2} \qquad (3.20)$$

from 3.13 it is evident that $\beta_0^2 = \left[1 - \frac{1}{\gamma^2}\left(1+K^2\right)\right] \Rightarrow \frac{1}{1-\beta_0^2} = \frac{\gamma^2}{\left(1+K^2\right)}$, then

$$\omega_0 = \frac{2\gamma^2}{\left(1+K^2\right)}\omega_u = \frac{2\gamma^2}{\left(1+K^2\right)}\frac{2\pi}{\lambda_u}c = \frac{2\gamma^2}{\left(1+K^2\right)}k_u c \qquad (3.21)$$

where $k_u = 2\pi/\lambda_u$

That expressed in terms of wavelength gives:

$$\lambda_u = \frac{2\pi}{\omega_0}c = 2\pi c\frac{\lambda_u}{2\pi c}\frac{1+K^2}{2\gamma^2} = \frac{\lambda_u}{2\gamma^2}(1+K^2) \qquad (3.22)$$

It is easy to notice that the dimension scale λ_u is reduced by a factor γ^2 exploiting the relativistic effects.
The emission bandshape can be easily calculated: radiation is emitted as a pulse train composed by $\frac{\omega' \Delta t'}{2\pi} = N'$ periods. The spectrum of such a structure is the well known "sinc ^{2}" function [12]:

$$f(\omega) \propto \left[\frac{sin^2(\vartheta/2)}{(\vartheta/2)^2} \right] \tag{3.23}$$

where $\vartheta = 2\pi N \frac{\omega_0 - \omega}{\omega_0}$ is the so-called "detuning parameter".
The bandshape for the undulator emission is shown in fig. 3.2.

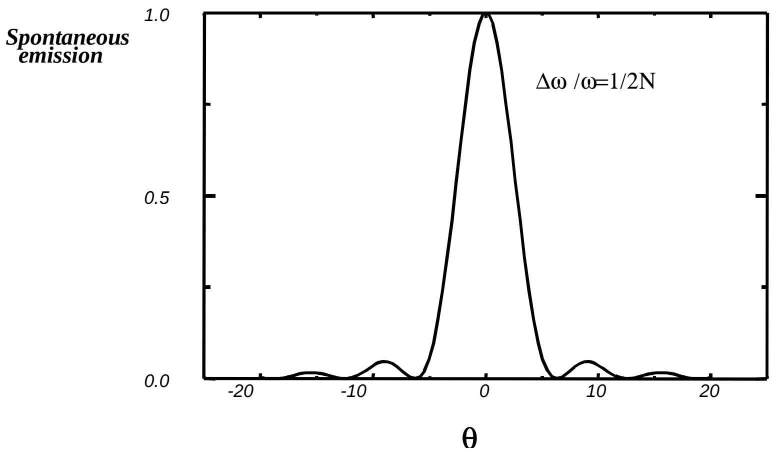

Fig. 3.2: Undulator emission bandshape

The emission of the undulator is not composed by a single line, at the central frequency, but is composed by a finite width band. This can be interpreted as a line broadening effect: in this case we are looking at the homogeneous broadening effect, due to the dynamic of the emission process, i.e. the finite transit time of the electron inside the undulator, that is equal for all electrons in the electron beam, independently on their physical properties, like energy, momentum and trajectory. This is analogous to the line broadening caused by finite lifetime in atomic systems.
In the electron beam the difference in energy, momentum and trajectory of the different electrons will cause a further broadening of the emission line, that can be classified as "inhomogeneous". The latter can be compared to inhomogeneous broadening in a gas emission due to doppler shift of the single atom emission.

4. Synchrotron radiation stimulated emission

Up to now we have seen what happens when an electron passes through a magnetic undulator. In such a process only a field is present: the static magnetic field of the undulator (that in the electron reference frame becomes an electromagnetic field). Let us now consider what happens when other EM modes are present during the interaction: we will observe the emission properties of such a system and the variations of the modes intensity during the process.

Let us consider an EM mode copropagating with the electrons inside the undulator. We want to calculate the rate of energy exchange between the electrons and the EM field.

Electrons oscillate inside the undulator in the transverse plane *xz* with period λ_u . In order to obtain energy exchange between the electrons and the EM field it is necessary to have synchronism between the transverse oscillations of the electrons and the oscillations of the Electric field of the copropagating EM wave. This will happens if the electron, after one undulator period, will find the electric field with the same phase. If we remind that the electron velocity $v_z < c$, it is evident that this condition can be fulfilled if the mean longitudinal electron velocity v_z is chosen in such a way that the electron performs a complete oscillation in the time needed for the light to cover an undulator period plus a wavelength. This condition can be expressed as follows:

- Being t_e the time needed for the electron to cover an undulator period λ_u
- Being t_p the time needed for the EM wave to cover the distance $\lambda_u + \lambda$

the synchronism condition is expressed by the equation $t_e = t_p$; if we remind that $t_e = \frac{\lambda_u}{v_z}$ and that $t_p = \frac{\lambda_u + \lambda}{v_f}$, where v_f is the phase velocity of the EM wave $v_f = \frac{\omega}{k}$, we obtain:

$$\frac{\lambda_u}{v_z} = \frac{\lambda_u + \lambda}{v_f} \tag{4.1}$$

if we define $k_u = 2\pi/\lambda_u$ eq. 4.1 can be rewritten as:

$$v_f \lambda_u = v_z(\lambda_u + \lambda) \Rightarrow \frac{v_f}{k_u} = v_z\left(\frac{1}{k_u} + \frac{1}{k}\right) = v_z\left(\frac{k + k_u}{k\, k_u}\right) \Rightarrow v_f\, k = v_z\left(k + k_u\right)$$

taking in mind that $v_f = \frac{\omega}{k}$ and $v_z = \beta_z c$ we obtain:

$$\frac{\omega}{c} = \beta_z\left(k + k_u\right) \tag{4.2}$$

this equation, that defines the so called "beam line", describes the points of the (k, ω/c) space where the synchronism condition is fulfilled

It is now necessary to consider the dispersion relation of the structure where the interaction takes place: *ω/c=f(ω).* The intersection between the curve representing the dispersion relation and the beam line in the (*k*, *ω/c*) space gives the emission frequency for the given undulator at the given beam energy. If the interaction occurs in vacuum, the dispersion relation is linear: $\omega/c=k$ and the emission frequency can be easily deduced from the intersection of this line with the beam line, as shown in fig. 4.1.

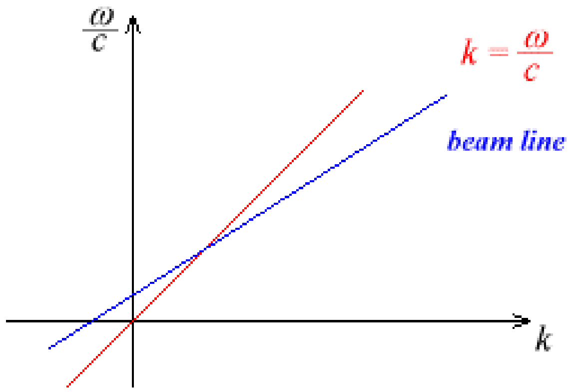

Fig. 4.1 : representation of the beam line and of the vacuum dispersion relation

The analytic solution is easily deduced solving the linear system:

$$\begin{cases} \dfrac{\omega}{c} = k \\ \dfrac{\omega}{c} = \beta_z (k + k_u) \end{cases} \Rightarrow k(1-\beta_z) = k_u \beta_z \quad \text{then, if we remind that } \gamma_z = \frac{1}{\sqrt{1-\beta_z^2}}$$

$$\frac{\omega}{c} = k = k_u \frac{\beta_z}{1-\beta_z} = k_u \frac{\beta_z}{1-\beta_z}\frac{1+\beta_z}{1+\beta_z} = k_u \frac{\beta_z(1+\beta_z)}{1-\beta_z^2} = k_u \beta_z \gamma_z^2 (1+\beta_z) \qquad (4.3)$$

Eq. 4.3 for relativistic electrons ($\beta \sim 1$) becomes:

$$\frac{\omega}{c} \approx 2\gamma_z^2 k_u \qquad (4.4)$$

We immediately notice that this formula is identical to eq. 3.21, giving the central spontaneous emission frequency for an undulator. In terms of wavelength we have (like in eq. 3.22)

$$\lambda = \frac{\lambda_u}{2\gamma^2}(1+K^2) \qquad (4.5)$$

It is easy to verify that synchronism condition is fulfilled also if in the time t_e the EM wave slips over the electron by a quantity equal to 2 or more wavelengths. The more general relation is then:

$$t_p = \frac{\lambda_u + n\lambda}{v_f} \qquad \text{with } n \text{ integer, greater than zero} \qquad (4.6)$$

Following the same calculation performed for the case n=1 it is possible to derive the general formulation for the beam line:

$$\frac{\omega}{c} = \beta_z \left(k + nk_u\right) \qquad (4.7)$$

n is the so called "harmonic number" and the intersection of the different beam lines characterized by different values of n with the dispersion relation, gives the emission frequencies for the fundamental (n =1) and the higher harmonics (n > 1), as shown in Fig. 4.2.

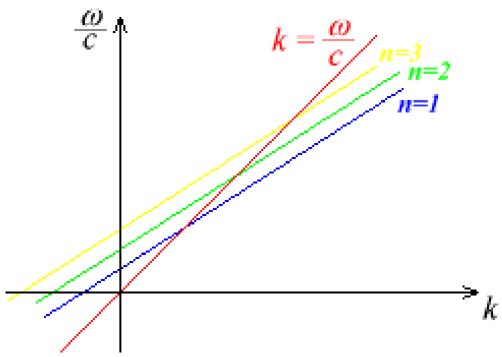

Fig. 4.2 : representation of the harmonics beam lines and of the vacuum dispersion relation

Then, it is possible to modify the emission frequency by changing:

- The electron energy
- The undulator period
- The K parameter, proportional to the undulator magnetic field.

4.1 GAIN

In order to calculate the achievable gain in the interaction between the electrons and the EM wave, we start calculating the energy variation of the electron: given $\gamma = \frac{E}{m_0 c^2}$ the energy variation can be expressed by:

$$\frac{d\gamma}{dt} = \frac{e}{m_0 c^2} \vec{E} \bullet \vec{v} \qquad (4.8)$$

Only the x and z components of the electron velocity are different from zero. In the EM wave the electric field is transverse, so that $\vec{E} \bullet \vec{v} = E_T v_x$; moreover we have:

$$\begin{cases} v_x = -\frac{\sqrt{2}cK}{\gamma} sin\left(\frac{2\pi}{\lambda_u} z\right) \\ E_T = E_0 \cos(\omega t - kz + \phi) \end{cases} \quad \text{where } \phi \text{ is the EM wave phase} \qquad (4.9)$$

with a simple substitution we obtain:

$$\begin{aligned} \frac{d\gamma}{dt} &= \frac{e}{m_0 c^2}\left(-\frac{\sqrt{2}cK}{\gamma} sin\left(\frac{2\pi}{\lambda_u} z\right) \bullet E_0 \cos(\omega t - kz + \phi)\right) = \\ &= -\frac{eE_0 K\sqrt{2}}{m_0 c\gamma} sin(k_u z)\cos(kz - \omega t + \phi) \end{aligned} \qquad (4.10)$$

using the simple trigonometric expression $sin\alpha \; sin\beta = \frac{1}{2}\left(sin(\alpha+\beta) + sin(\alpha-\beta)\right)$ we have:

$$\begin{aligned} \frac{d\gamma}{dt} &= -\frac{eE_0 K\sqrt{2}}{2m_0 c\gamma}\{sin[(k_u + k)z - \omega t + \phi] + sin[(k_u - k)z + \omega t - \phi]\} = \\ &= -\frac{eE_0 K\sqrt{2}}{2m_0 c\gamma}\{sin[(k + k_u)z - \omega t + \phi] - sin[(k - k_u)z - \omega t + \phi]\} = \\ &= -\frac{eE_0 K\sqrt{2}}{2m_0 c\gamma}\left(sin\psi^- - sin\psi^+\right) \end{aligned} \qquad (4.11)$$

with $\psi^{\pm} = sin(k \mp k_u)z - \omega t + \phi$

both terms are rapidly oscillating with t, but if ω is chosen close to the resonance frequency it is possible to neglect the second term, because it will oscillate more rapidly, giving then a negligible contribution after averaging over time. Thus, approximating $\psi = \psi^-$ we can write:

$$\frac{d\gamma}{dt} = -\frac{eE_0K\sqrt{2}}{2m_0c\gamma} sin\psi \qquad (4.12)$$

with $\psi = \left(k + \frac{2\pi}{\lambda_u}\right)z - \omega t + \phi$ (4.13)

It is now possible to write the time derivative of the function ψ:

$$\begin{cases} \frac{d\psi}{dt} = -\omega + \left(k + \frac{2\pi}{\lambda_u}\right)\frac{dz}{dt} \\ \frac{d^2\psi}{dt^2} = \left(k + \frac{2\pi}{\lambda_u}\right)\frac{d^2z}{dt^2} \end{cases} \qquad (4.14)$$

let us remind that $v^2 = \left(\frac{dz}{dt}\right)^2 + \langle v_x^2 \rangle = c^2\left(1 - \frac{1}{\gamma^2}\right)$

the equation expressing *dγ/dt* must now be coupled with the expression for *dz/dt*, that can be simplified averaging over *z* the longitudinal oscillation, that is negligible when compared to the longitudinal velocity. The average over *z* of the $sin(k_u z)$ results in a factor ½, so that equation (4.9) for v_x becomes $v_x^2 = -\frac{2c^2K^2}{2\gamma^2}$, then:

$$\left(\frac{dz}{dt}\right)^2 = c^2\left(1 - \frac{1}{\gamma^2} - \frac{K^2}{\gamma^2}\right) = c^2\left(1 - \frac{1}{\gamma^2}\left(1 + K^2\right)\right) \qquad (4.15)$$

using the series expansion $\sqrt{1+x} = 1 + \frac{1}{2}x + o(x^2)$ we obtain:

$$\frac{dz}{dt} = c\sqrt{1 - \frac{1+K^2}{\gamma^2}} \approx c\left(1 - \frac{1+K^2}{2\gamma^2}\right) \qquad (4.16)$$

performing the time derivative:

$$\frac{d^2z}{dt^2} = c\frac{d\gamma/dt}{\gamma^3}\left(1 + K^2\right) \qquad (4.17)$$

substituting the value (4.12) for $d\gamma/dt$ (4.12) and the expression of $K = \frac{eB_0\lambda_u}{2\pi\sqrt{2}m_0c^2}$ we obtain:

$$\frac{d^2z}{dt^2} = -\frac{e^2E_0B_0\lambda_u}{4\pi\gamma^4(m_0c)^2}\left(1+K^2\right)sin\psi \tag{4.18}$$

let us remind that $\frac{d^2\psi}{dt^2} = \left(k+k_u\right)\frac{d^2z}{dt^2}$, so that:

$$\frac{d^2\psi}{dt^2} = -\left(k+k_u\right)\frac{e^2E_0B_0\lambda_u}{4\pi\gamma^4m_0^2c^2}\left(1+K^2\right)sin\psi \tag{4.19}$$

if we now define[1]:

$$\Omega^2 = \left(k+k_u\right)\frac{eE_0K}{\sqrt{2}\gamma^4m_0}\left(1+K^2\right) \tag{4.20}$$

It is possible to write the famous "pendulum equation" that describes the FEL dynamics [13, 14]:

$$\frac{d^2\psi}{dt^2} = -\Omega^2 sin\psi \tag{4.21}$$

The Gain per single pass is defined as the relative energy variation of the radiation, i.e.:

$$G = \frac{\Delta W_p}{W_p} \tag{4.22}$$

it is easy to calculate the value of ΔW_p, because the energy increase of the EM mode corresponds to the energy loss of the electrons, thus:

$$\Delta W_p = -m_0c^2\Delta\gamma \tag{4.23}$$

we can then write:

$$G = -m_0c^2\frac{\Delta\gamma}{W_p}$$

The variation of γ can be expressed in terms of ψ; if $\Delta\gamma << \gamma$ we have:

[1] It is valid if γ= constant over the single pass, i.e. in low Gain conditions

$$\begin{cases} \Delta\left(\frac{dz}{dt}\right) = \frac{c\left(1+K^2\right)}{\gamma^3}\Delta\gamma \\ \Delta\left(\frac{d\psi}{dt}\right) = \left(k+k_u\right)\Delta\left(\frac{dz}{dt}\right) \end{cases} \tag{4.24}$$

we obtain then:

$$\Delta\gamma = \frac{\gamma^3}{c\left(1+K^2\right)\left(k+k_u\right)}\ \Delta\left(\frac{d\psi}{dt}\right) \tag{4.25}$$

and gain can be written as:

$$G = -m_0 c^2 \frac{\gamma^3}{c\left(1+K^2\right)\left(k+k_u\right)}\ \Delta\left(\frac{d\psi}{dt}\right)\frac{1}{W_p} \tag{4.26}$$

If in the electron beam there are *N* electrons, corresponding to a beam current *I*,

$$N = \frac{IL}{ec\beta_z} \tag{4.27}$$

considering *N* times the mean energy exchanged by each electron we have:

$$G = -\frac{IL}{e\beta_z}\frac{m_0\gamma^3}{\left(1+K^2\right)\left(k+k_u\right)}\left\langle \Delta\left(\frac{d\psi}{dt}\right)\right\rangle\frac{1}{W_p} \tag{4.28}$$

To obtain the value for $\left\langle \Delta\left(\frac{d\psi}{dt}\right)\right\rangle$ it is possible to perform a perturbative analysis of the pendulum equation, that is beyond the task of this paper. Here we report only the final results of the gain calculations:

$$G = \pi\frac{K^2\left(1+K^2\right)}{\left(\beta_z\gamma\right)^5}\frac{\left(\lambda_u N\right)^3}{\Sigma_L}\frac{I}{I_0}\left(k+k_u\right)\frac{d}{d\theta}\left[\left(\frac{sin\theta/2}{\theta/2}\right)^2\right] \tag{4.29}$$

where $\theta = 2\pi n\frac{\omega - \omega_0}{\omega_0}$ is the so-called detuning parameter, I_0 is the Alfven current, given by $I_0 = \frac{ec}{r_0} \approx 17$ kA and Σ_L is the laser mode section. In vacuum Σ_L is the section of the gaussian mode into the resonator, while inside a waveguide, for a generic TE_{mn} mode, is expressed by:

$$\Sigma_L = \frac{\left(\frac{m}{a}\right)^2 + \left(\frac{n}{b}\right)^2}{\left(\frac{n}{b}\right)^2} \frac{ab}{4} \sigma \tag{4.30}$$

where a and b are the waveguide transverse dimensions and $\sigma = 1$ if both m and n are different from zero, $\sigma = 2$ otherwise.

It is easy to notice that the gain depends on many parameters. In order to simplify the expression f the gain, we can perform some approximations:

1. $k_u << k$
2. operation at resonance: $\left(\lambda_0 = \frac{\lambda_u}{2\gamma^2} \left(1 + K^2\right) \right)$

With these approximations the expression for the gain is:

$$G = 2\pi \frac{K^2}{\beta^5{}_z \gamma^3} \frac{\lambda_u{}^2 N^3}{\Sigma_L} \frac{I}{I_0} \frac{d}{d\theta} \left[\left(\frac{\sin \theta/2}{\theta/2} \right)^2 \right] = 4\pi^2 \frac{K^2 \lambda_u{}^2 N^3}{\beta^5{}_z \gamma^3} \frac{I}{I_0} \frac{d}{d\theta} \left[\left(\frac{\sin \theta/2}{\theta/2} \right)^2 \right] \tag{4.31}$$

Gain is proportional to the beam current intensity, to the cube of the number of undulator periods, to the square of the undulator parameter K, that is in turn proportional to the undulator magnetic field intensity, while it is inversely proportional to the cube of the energy γ.
Once the machine is built, it is usually impossible to change the values of γ and N , so that the only parameters that can be exploited to increase the gain while operating are the beam current I and the undulator parameter K, that can be modified by varying the undulator poles distance or, in electromagnetic undulators, by varying the current generating the magnetic field.
The electron beam current is then a critical parameter for the FEL operation, and we need to obtain high currents to achieve values of the gain compatibles with laser emission. The dependence on $1/\gamma^3$ is one of the reasons that makes so difficult to build short wavelength FELs (λ is proportional to $1/\gamma^2$).

4.2 EFFICIENCY

Let us now consider once more equation 4.31 for the small signal gain, reported in fig. 4.3:

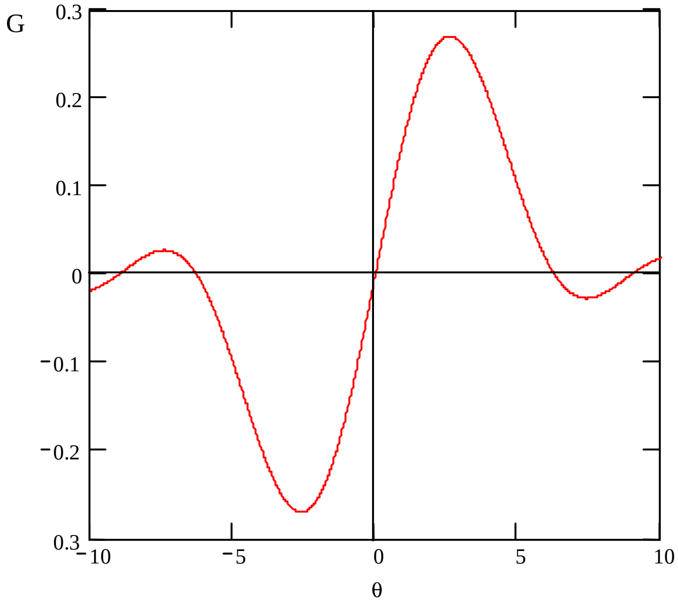

Fig. 4.3 : Small signal gain as a function of the detuning parameter θ

If due to the interaction the mean energy of the electrons decreases so much that the detuning parameter $\theta = 2\pi N \frac{\omega - \omega_0}{\omega_0}$ is no more in the positive branch of the gain curve, the FEL emission process stops. This happens when $\Delta\theta > 2\pi$, then the maximum value of $\Delta\theta$ compatible with FEL emission is:

$$\Delta\theta \leq 2\pi \tag{4.32}$$

which corresponds to a maximum energy variation of the electron beam ΔE, that can be deduced as follows: the FEL emission frequency ω_0 is proportional to the square of the electron beam energy: $\omega_0 \propto E^2 \Rightarrow \Delta\omega_0 \propto 2E\Delta E$ then

$$\frac{\Delta E}{E} = \frac{1}{2}\frac{\Delta\omega_0}{\omega_0} = \frac{\Delta\theta}{4\pi N} \leq \frac{1}{2N} \tag{4.33}$$

This is the maximum mean energy variation of the electron beam allowed in order to continue emitting, i.e. the maximum theoretical efficiency for the FEL emission process:

$$\eta \leq \frac{1}{2N} \tag{4.34}$$

This equation is known as "Renieri limit" for the FEL efficiency [10].
It is possible to notice that while gain increases like N^3, the efficiency decreases with N. This can be understood if we remind the spectral characteristics of the undulator emission: if the number of periods (i.e. the length) of the undulator is increased, an hypothetical observer placed along the undulator axis will see a longer light pulse, because the time duration of the pulse is the time needed to the light to pass through the longer undulator, and the emission will take place on a narrower spectral band. Being longer the interaction time, also the gain will be higher, but being the gain curve proportional to the derivative of the emission band, with a narrower gain curve the electrons losing energy will go out of tune more rapidly, thus reducing the efficiency of the process.
One possible solution to the Renieri efficiency limit consists in trying to "follow" the electron energy loss, smoothly changing the parameters of the FEL along the undulator, in order to change the interaction frequency while the electron loses energy. This is the so-called undulator tapering, i.e. the variation of the undulator gap along the z axis, in order to change the K parameter. This way the central emission frequency will move together with the electron energy "motion" under the Gain curve, increasing the total efficiency
Another possibility is related to energy recovery: it is possible to collect the already unuseful electrons after the interaction and recover their energy, that can be used to accelerate new electrons.

4.3 FEL LINE BROADENING

The expression of the gain has been obtained considering a monoenergetic electron beam, with zero displacement and zero angular divergence from the undulator axis. In a real beam it is not possible to neglect the effects of energy spread and beam divergence.
These deviations from the ideal conditions will cause a broadening of the emission band, much alike an "inhomogeneous broadening", that causes the reduction of the gain and the displacement of the central emission frequency. The latter can be expressed by [15]:

$$\omega_0 = \frac{4\pi c\gamma^2}{\lambda_u} \frac{1}{\left(1 + K^2 + \alpha^2\gamma^2\right)} \tag{4.35}$$

The correction term $\alpha^2\gamma^2$, is due to off-axis electrons (α is the deviation angle from the undulator axis).
It is possible to write the single contributions of energy and angular dispersion to inhomogeneous broadening [10]:

$$\left.\frac{\Delta\omega}{\omega}\right|_i = \left[\left(\frac{\Delta\omega}{\omega}\right)_\varepsilon^2 + \left(\frac{\Delta\omega}{\omega}\right)_x^2 + \left(\frac{\Delta\omega}{\omega}\right)_y^2\right]^{1/2} \quad (4.36)$$

If we define σ_ε , the standard deviation of the energy distribution of the electron, it can be proved that:

$$\left.\frac{\Delta\omega}{\omega}\right|_\varepsilon = 2\sigma_\varepsilon \quad (4.37)$$

Moreover, if we consider the undulator field being not perfectly sinusoidal, with the presence of sextupole terms h_x and h_y, and an electron beam characterized by transverse emittances ε_x and ε_y , the broadening terms dependent on the coordinates can be written as:

$$\left.\frac{\Delta\omega}{\omega}\right|_{x,y} = \sqrt{2\left|h_{x,y}\right|}\,\frac{\gamma\varepsilon_{x,y}}{\lambda_u}\,\frac{K}{1+K^2} \quad (4.38)$$

In order to evaluate the effect of the inhomogeneous broadening, it is necessary to compare it with the homogeneous broadening term, due to the finite transit time of the electrons inside the undulator. It is then possible to define the so called inhomogeneous broadening normalized parameters:

$$\mu_{\varepsilon,x,y} = \frac{\left(\frac{\Delta\omega}{\omega}\right)_{e,x,y}}{\left(\frac{\Delta\omega}{\omega}\right)_0} \quad (4.39)$$

If $\mu_{\varepsilon,x,y} << 1$, the FEL is operating in homogeneous broadening regime:

$$\left.\frac{\Delta\omega}{\omega}\right|_{TOT} = \left(\frac{\Delta\omega}{\omega}\right)_0 \left[1 + \mu_\varepsilon + \mu_x + \mu_y\right] \quad (4.40)$$

where the term $\mu_\varepsilon+\mu_x+\mu_y$ describes the deviation from the homogeneous regime. The effects of the inhomogeneous broadening can be seen in fig. 4.4:

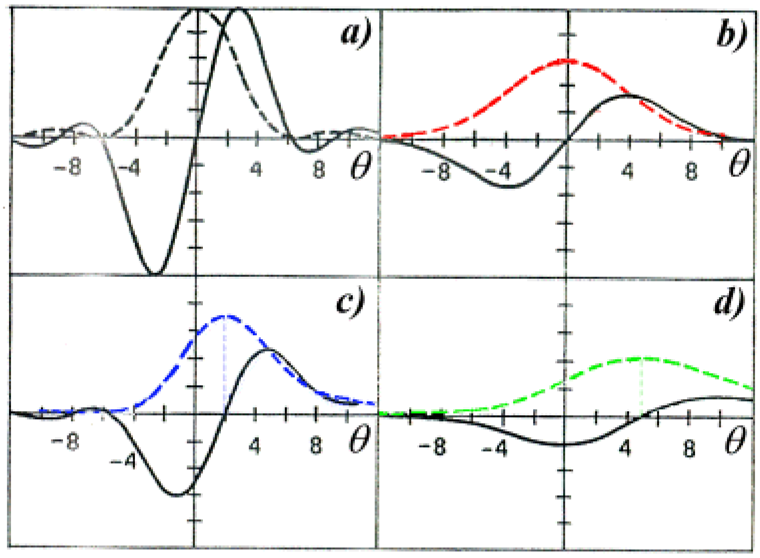

Fig. 4.4 Emission band (- - -) and gain (___) in the homogeneous broadening regime (a), inhomogeneous broadening caused by energy spread $\mu_\varepsilon = 1$ (b) , inhomogeneous broadening caused by angular dispersion along x $\mu_x=1$ (c), inhomogeneous broadening caused by energy spread and angular dispersion $\mu_\varepsilon = \mu_x = \mu_y = 1$ (d) [10]

It is possible to perform a numerical analysis of the effects of inhomogeneous broadening on the peak value of the gain. Such an analysis gives:

$$G_{\max} = \frac{G_0}{\left(1+1.7\mu_\varepsilon^2\right)+\left(1+\mu_x^2\right)+\left(1+\mu_y^2\right)} \tag{4.41}$$

where G_0 is the value of the gain calculated taking into account homogeneous broadening only.

It is possible to verify that increasing the number of undulator periods N, also the value of the parameters μ_i increases, increasing thus the inhomogeneous broadening.

When the FEL operates with a radiofrequency driven electron accelerator, a further source of inhomogeneous broadening appears: the bunch structure of the electron beam is reflected in an analogous structure in the emitted light, and the laser pulse is then composed by a train of light pulses. In order to obtain the interaction between the electron bunches and the light pulses propagating back and forth inside the optical resonator, there must be superposition between electrons and light, i.e. the light pulse should find, after a round trip, another electron bunch entering the cavity. This means that the time distance between two (or more than two) electron bunches must be equal to the round trip time for the light pulse in the resonator. This distance, for a RF accelerator, is the radiofrequency period T, so that the "matching" condition can be written as:

$$\frac{2L}{c} = nT \qquad (4.42)$$

where n is a positive integer; n =2 means that the light pulse generated by the i-th electron bunch, after a round trip will not find the (i+1)-th electron bunch, but the (i+2)-th (there is a "jump" of 1 bunch), and so on for n=3,4,…
Anyway, also if this condition is fulfilled, we must be aware that the light velocity in vacuum is always greater that the electron velocity, thus along the undulator period the light pulse slips over the electron bunch by the amount:

$$\Delta = N\lambda \qquad \text{the so called "slippage length"} \qquad (4.43)$$

This relation follows immediately if we remind that the synchronism condition for FEL operation requires that the light pulse passes over the electron bunch by the a wavelength λ after an undulator period λ_u, therefore, after N periods we obtain the expression for Δ.
The existence of the slippage produces some effects:

1. electrons and photons are not perfectly superimposed over the whole interaction time
2. due to the fact that the forward part of the pulse overcomes the electron bunch, the rear part of the pulse will interact for a longer time, experiencing an higher gain. This asymmetry results in the shift of the center of the pulse, that seems to be slowed (fig. 4.5). This effect, called "lethargy", produces as an effect the fact that in order to obtain the matching between electron bunches and light pulses, the cavity length must be reduced of a small length dL.

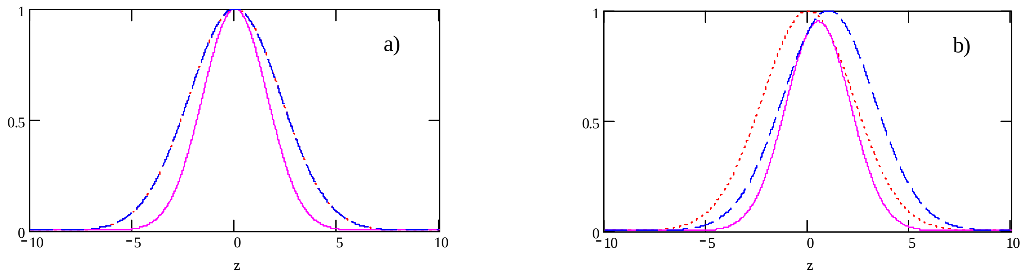

Fig. 4.5 Lethargy mechanism: (a) when the interaction starts the electron bunch (···) and the light pulse are superimposed, and their interaction produces radiation (—); (b) the light pulse (---) begins overcoming the electron bunch (···) the interaction generates new radiation (—) that appears to be "retarded" respect to the original pulse position.

A possible way to avoid the slippage consists in slowing down the light pulse, making use of a waveguide resonator. Due to dispersion relation of the waveguide, the light velocity inside the waveguide can be slower then c, and it is possible to make it equal to the mean longitudinal velocity of the electrons.
The effect of the slippage can be evaluated introducing the parameter:

$$\mu_c = \frac{N\lambda}{\sigma_z} \tag{4.44}$$

where σ_z is the normalized standard deviation of the longitudinal distribution of the electrons.

μ_c is called "normalized slippage", and, like for the other μ parameters, an expression relates the maximum gain to the value of μ_c:

$$G_{max} = \frac{G_0}{1 + \frac{\mu_c}{3}} \tag{4.45}$$

So there will be a gain reduction due to the non perfect superimposition of the electron pulse and the light pulse. This confirms the advantages of using a waveguide resonator when dealing with long wavelength emission, because it is possible to "slow down" the light pulse in the waveguide and operate at "zero slippage" condition.

5. Free Electron Laser components

In the typical Free Electron Laser electrons are accelerated by an electron accelerator, then, by means of a transport channel, they are injected inside the undulator, where the FEL interaction occurs inside an optical resonator. The emitted light is then extracted from the resonator and delivered to the detection section.
The typical layout for an FEL is reported in Fig. 5.1

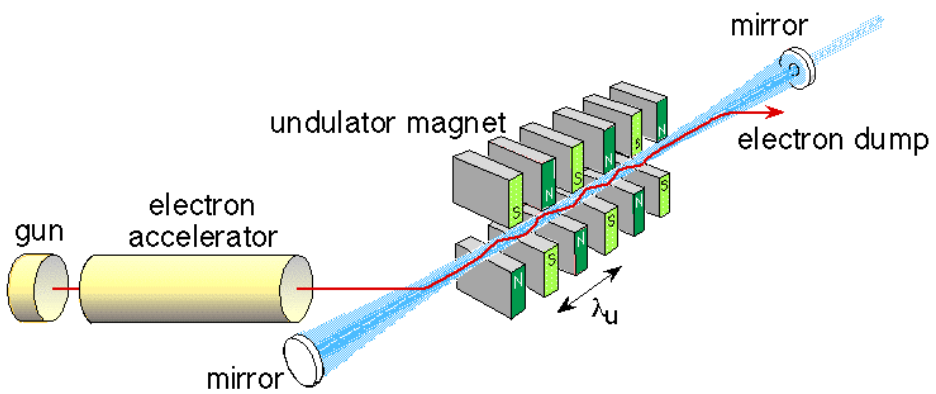

Fig. 5.1: Typical FEL layout

The electron accelerator is probably the most important component of the FEL: most of the performance of the laser depends on the characteristics of the electron beam.

- The laser emission wavelength depends on the electron energy γ :
- The gain strongly depends on the current in the e-beam
- The gain is strongly affected by the e-beam quality: we have seen that energy spread ΔE and angular deviation in the beam induces gain reduction.

In order to better define the e-beam quality it is possible to introduce the so-called "beam emittance": let us consider a single transverse coordinate, say *x*, and let's have a look to the phase space (*x*,*x'*). The ideal particle, with zero displacement and deviation from the axis would lay in the (0,0) point. In a real beam each electron will have its own position and divergence, and will be represented by a point in the phase space *x*-*x'*. We can define the beam emittance ε_x the area of the ellipse containing 95% of the particles, as shown in fig. 5.2

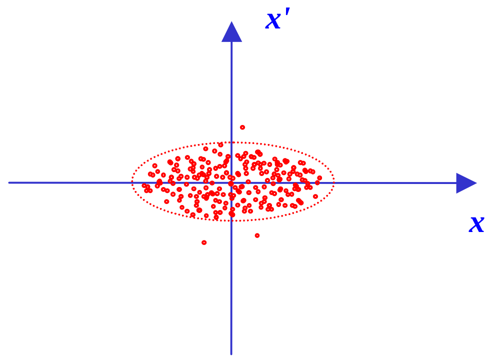

Fig. 5.2: transverse phase space and definition of the emittance

The parameters ΔE, representing the electrons energy spread, together with the emittances ε_x and ε_y define the beam quality, determining the central frequency shift, the broadening of the emission and the reduction of the gain
The choice of the accelerator depends on the required emission spectral region: looking at the FEL emission wavelength it is easy to notice that tunability can be exploited changing the value of K, but the emission spectral region is mainly determined by the electron energy γ, that is a design parameter and usually cannot be changed during operation.
The choice of the accelerator depends also on the required characteristics of the output radiation: the temporal structure of the output radiation depends on the temporal structure of the e-beam, so if continuous emission is needed we must use an electrostatic accelerator, like a Van der Graaf generator, a Cockroft-Walton accelerator or a Tandem accelerator.
These accelerators can produce a continuous electron beam using charge recovery technique, i.e. the e-beam is recirculated from the output of the undulator back into the accelerator. Practical limitations arise from the size of such devices, and the maximum

electron energy is limited to a few MeV, corresponding to an emission in the region between the mm-wave and the FIR.
If a pulsed radiation structure is acceptable, Radio Frequency Linacs are probably the best choice.
In such a device an electromagnetic Radio Frequency field is used to accelerate the electrons up to high energy.
The electrons produced by such a device are composed of a sequence of macropulses, with a time duration of the order of μs, each one composed by a series of micropulses, of time duration of a few ps, separated by the RF period (fig. 5.3).

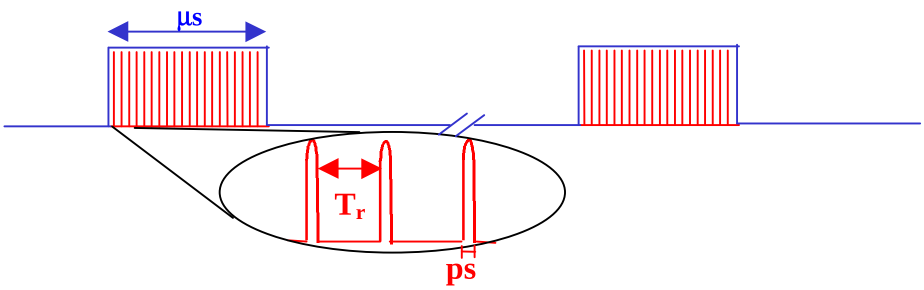

Fig. 5.3 : Time structure of a radiofrequency accelerator electron emission

The next component of the system is the electron transport channel. It is usually composed by a metal pipe, in high vacuum conditions, with magnetic elements to steer and focus the electron beam.
Simple dipole coils are used to deflect the beam in order to perform position and angle corrections. This allow to inject electrons in the undulator perfectly "on-axis". In order to avoid the natural tendency of the electron bunch to expand, magnetic quadrupoles are generally used. Quadrupoles act on the e-beam exactly in the same way a lens acts on a light beam. The matrix formalism used for magnetic lenses can be applied to standard optics.
Other magnetic elements can be used to perform various operations on the e-beam.

The next element of the system is the magnetic undulator. The undulator can be built using permanent magnets or using electromagnetic devices. There are two main geometries: the planar undulator and the helical undulator.
In building a planar undulator the task is to obtain a field as close as possible to a purely sinusoidal field. In order to obtain such a field configuration it is generally used the so-called "Halbach configuration", that makes use of 4 magnets per pole per period (fig. 5.4).

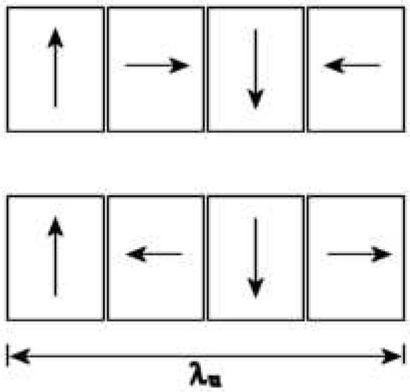

Fig. 5.4: Halbach configuration for undulator magnets

Single magnets field should be measured with care, and then the magnets must be sorted by means of a dedicated software. When the undulator is built, the field map can be measured and, if necessary, it is possible to insert some thin ferromagnetic foils in the right position in the undulator, in order to correct the small deviations from the sinusoidal behaviour (shimming technique).

The optical resonator acts exactly like in a conventional laser. One more constraint appears when using a RF accelerator as electron source: the resonator length must be chosen in order to allow superposition of the light pulse with subsequent electron bunches. This means that the round trip transit time for the radiation must be an integer multiple of the RF period: in vacuum this can be expressed by eq. 4.42: $\frac{2L}{c} = nT$

The radiation is finally extracted from the resonator and sent to the detection area, where performances of the FEL are measured and experiments can be performed using the FEL radiation. The characteristics of the detectors depend on the FEL parameters. Usually FELs are able to deliver high peak power, so that detectors must be chosen taking in mind this characteristic of the output radiation.

6. Conclusions

Principles of operation of the Free Electron Laser have been presented. The present technology allows the design and realization of FELs emitting in a wide spectral range, from the mm-wave region up to the X-ray region. Being the FEL a complex and expensive machine, it is not convenient to use it in a region already covered by conventional lasers. From the above analysis it appears logical to limit the interest to the extreme zones of the spectrum, i.e. the mm-wave/FIR region, beyond the possibilities of conventional free electron devices, where interesting applications can be found in biology, solid state and molecular physics, and where the costs related to the FEL realization are not too high. At the other end of the spectrum we find the x-ray region, where a high brightness coherent source, emitting very short pulses would be of enormous importance to many application in different research fields, and could justify the very high costs and the technical difficulties related to the realization of such a device. At present there are 2 machines under construction worldwide, and a third one is being designed in Italy. The realization of such a machine will improve dramatically the experimental possibilities respect to the actual situation, that makes use of synchrotron radiation in the X-ray region.

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